

Averages of alpha-determinants over permutations

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Abstract

We show that certain weighted average of the α -determinant of a kn by kn matrix of the form $A \otimes \mathbf{1}_{1,k}$, the Kronecker product of a kn by n matrix A and 1 by k all one matrix $\mathbf{1}_{1,k}$, over permutations of kn letters is reduced to the k -wreath determinant of A up to constant. The constant is exactly given by the modified content polynomial for the Young diagram (k^n) . As a corollary, we give a ‘determinantal’ formula for certain functions on the symmetric groups which are invariant under the left and right translation by a Young subgroup, especially the values of the Kostka numbers for rectangular shapes with arbitrary weight. This corollary gives a generalization of the formula of irreducible characters of the symmetric group for rectangular shapes due to Stanley.

1 Introduction

The α -determinant of an N by N square matrix $A = (a_{ij})$ is defined as a parametric deformation of the usual determinant as

$$\det_{\alpha} A := \sum_{\sigma \in \mathfrak{S}_N} \alpha^{\nu(\sigma)} a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(N)N},$$

where α is a complex parameter and $\nu(\sigma)$ for a permutation $\sigma \in \mathfrak{S}_N$ is defined to be N minus the number of disjoint cycles in σ . By definition, we see that

$$\det_{-1} A = \det A, \quad \det_1 A = \text{per } A, \quad \det_0 A = a_{11} a_{22} \cdots a_{NN},$$

where $\text{per } A$ is the *permanent* of A . It is Vere-Jones [8] who first introduce such a parametric deformations, which he called the α -*permanent*. Here we adopt the modified definition and terminology by Shirai and Takahashi [5]. The α -determinant is multiplicative only if $\alpha = -1$.

Let $P(\sigma) = (\delta_{i\sigma(j)})$ be the permutation matrix for a permutation $\sigma \in \mathfrak{S}_N$. The sum

$$\sum_{\sigma \in \mathfrak{S}_k} \det_{\alpha}(AP(\sigma)) \tag{1.1}$$

is a polynomial in α which is divisible by $(1 + \alpha) \cdots (1 + (k - 1)\alpha)$ for a given N by N matrix A . Here we regard \mathfrak{S}_k as a subgroup of \mathfrak{S}_N consisting of permutations which do not move the $N - k$ letters $k + 1, k + 2, \dots, N$. This fact is used to show that the α -determinant is weakly alternating when α is a reciprocal of a negative integer in the sense that $\det_{-1/k} A$ vanishes whenever more than k columns or rows in A are equal (Lemma 3.3). Based on this fact, we define the k -wreath determinant $\text{wrdet}_k A$ of a $kn \times n$ matrix A by

$$\text{wrdet}_k A := \det_{-1/k} (\overbrace{\mathbf{a}_1, \dots, \mathbf{a}_1}^k, \dots, \overbrace{\mathbf{a}_n, \dots, \mathbf{a}_n}^k),$$

where \mathbf{a}_j is the j -th column vector of A . This *recovers* the relative invariance

$$\text{wrdet}_k(AQ) = \text{wrdet}_k A (\det Q)^k$$

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with respect to the right translation by any n by n matrix Q [1].

In the extremal case where $k = N$, we can determine the sum (1.1) explicitly as

$$\sum_{\sigma \in \mathfrak{S}_N} \det_{\alpha}(AP(\sigma)) = \prod_{i=1}^{N-1} (1 + i\alpha) \cdot \text{per } A.$$

More generally, one can prove

$$\sum_{\sigma \in \mathfrak{S}_N} \chi^{\lambda}(\sigma) \det_{\alpha}(AP(\sigma)) = f^{\lambda} f_{\lambda}(\alpha) \text{Imm}_{\lambda} A \quad (1.2)$$

for any partition $\lambda \vdash N$. Especially we have

$$\sum_{\sigma \in \mathfrak{S}_N} \text{sgn } \sigma \det_{\alpha}(AP(\sigma)) = \prod_{i=1}^{N-1} (1 - i\alpha) \cdot \det A. \quad (1.3)$$

Here χ^{λ} is the irreducible character of \mathfrak{S}_N associated to λ , f^{λ} is the number of standard tableaux with shape λ , $f_{\lambda}(\alpha)$ is the modified content polynomial for λ and $\text{Imm}_{\lambda} A$ is the immanant of A associated to λ . The identity (1.2) is essentially equivalent to the result by Matsumoto and Wakayama [4] on the irreducible decomposition of the $U(\mathfrak{gl}_N)$ -cyclic submodule generated by a single polynomial $\det_{\alpha} X$. The structure of such cyclic module is the same for almost all values of α , but changes drastically when α is a reciprocal of a nonzero integer.

The purpose of the paper is to give an analog of (1.3) for the k -wreath determinant (Theorem 2.2). As corollaries of the main result, we also obtain a formula for certain \mathfrak{S}_{μ} -biinvariant functions on \mathfrak{S}_N , where \mathfrak{S}_{μ} is the Young subgroup of \mathfrak{S}_N associated with a partition $\mu \vdash N$. In particular, we get a formula for Kostka numbers with rectangular shape and arbitrary weight (Corollaries 5.1, 5.4). These corollaries give a generalization of the formula for irreducible characters of the symmetric groups associated to rectangular diagrams which is due to Stanley [6] (Corollary 5.2).

2 Weighted averages of alpha-determinants over permutations

Let n, k be positive integers. We define a linear map $\varpi_k: M_{kn,n} \rightarrow M_{kn}$ by $\varpi_k(A) := A \otimes \mathbf{1}_{1,k}$, where $M_{p,q}$ is the set of p by q complex matrices, $M_p = M_{p,p}$ is the set of square matrices of size p , $\mathbf{1}_{p,q}$ is the p by q all-one matrix, and \otimes denotes the Kronecker product of matrices

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \quad (A = (a_{ij}) \in M_{m,n}).$$

We note that ϖ_k commutes with the left translation, that is, $\varpi_k(PA) = P \varpi_k(A)$ for any $P \in M_{kn}$ and $A \in M_{kn,n}$. We also notice that

$$\varpi_k(A)P(g) = \varpi_k(A) \quad (2.1)$$

for any $A \in M_{kn,n}$ and $g \in \mathfrak{S}_k^n = \mathfrak{S}_{(k^n)}$. For a kn by n matrix $A \in M_{kn,n}$, the k -wreath determinant of A is defined by

$$\text{wrdet}_k A := \det_{-1/k} \varpi_k(A). \quad (2.2)$$

The 1-wreath determinant is the ordinary determinant: $\text{wrdet}_1 A = \det A$. See [1] for basic facts on the wreath determinants.

Example 2.1 ($n = k = 2$). For $A = (a_{ij}) \in M_{4,2}$, the 2-wreath determinant of A is

$$\begin{aligned} \text{wrdet}_2 A &= \det_{-1/2} \begin{pmatrix} a_{11} & a_{11} & a_{12} & a_{12} \\ a_{21} & a_{21} & a_{22} & a_{22} \\ a_{31} & a_{31} & a_{32} & a_{32} \\ a_{41} & a_{41} & a_{42} & a_{42} \end{pmatrix} \\ &= \frac{1}{4} \left\{ a_{11}a_{21}a_{32}a_{42} + a_{12}a_{22}a_{31}a_{41} \right\} \\ &\quad - \frac{1}{8} \left\{ a_{11}a_{22}a_{31}a_{42} + a_{11}a_{22}a_{32}a_{41} + a_{12}a_{21}a_{31}a_{42} + a_{12}a_{21}a_{32}a_{41} \right\}. \end{aligned}$$

We can express $\text{wrdet}_2 A$ as a sum of products of minor determinants of A as

$$\text{wrdet}_2 A = \frac{1}{8} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix} + \frac{1}{8} \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$

which apparently shows the relative invariance $\text{wrdet}_2(AQ) = \text{wrdet}_2 A (\det Q)^2$ for $Q \in M_2$.

For a partition λ , $f_\lambda(\alpha)$ is the *modified content polynomial* for λ

$$f_\lambda(x) = \prod_{(i,j) \in \lambda} (1 + (j - i)x),$$

where we identify λ with its corresponding Young diagram. For instance, we have

$$f_{(N)}(x) = \prod_{i=1}^{N-1} (1 + ix), \quad f_{(1^N)}(x) = \prod_{i=1}^{N-1} (1 - ix)$$

for any positive integer N . It is notable that

$$\det_\alpha \mathbf{1}_N = \sum_{\sigma \in \mathfrak{S}_N} \alpha^{\nu(\sigma)} = f_{(N)}(\alpha), \quad (2.3)$$

where $\mathbf{1}_N = \mathbf{1}_{N,N}$.

Our goal is to prove the

Theorem 2.2. *For each positive integer k , the equality*

$$\sum_{\sigma \in \mathfrak{S}_{kn}} \left(-\frac{1}{k}\right)^{\nu(\sigma)} \det_\alpha(\varpi_k(A)P(\sigma)) = f_{(k^n)}(\alpha) \text{wrdet}_k A \quad (2.4)$$

holds.

When $k = 1$, the theorem is reduced to the equality (1.3).

3 Proof of the theorem

For later use, we put $\mathbb{I}_{n,k} = I_n \otimes \mathbf{1}_{k,1}$. We postpone the proofs of the lemmas used in this section to §4.

3.1 Reduction

To prove the theorem, we need the characterization of the k -wreath determinant.

Lemma 3.1 (Corollary 5.8 in [1]). *Suppose that a function $f: M_{kn,n} \rightarrow \mathbb{C}$ satisfies the following conditions.*

(W1) *f is multilinear in row vectors.*

(W2) *$f(AQ) = f(A) (\det Q)^k$ for any $Q \in M_n$.*

(W3) $f(P(g)A) = f(A)$ for any $g \in \mathfrak{S}_k^n$.

Then f is equal to the k -wreath determinant wrdet_k up to constant multiple.

To determine the constant factor explicitly in our discussion below, the formula

$$\text{wrdet}_k \mathbb{I}_{n,k} = \det_{-1/k}(I_n \otimes \mathbf{1}_k) = \left(\frac{k!}{k^k}\right)^n$$

is useful (see Lemma 4.6 in [1]).

Let $F(\alpha; A)$ be the left-hand side of (2.4). Since the multilinearity of $F(\alpha; A)$ in row vectors of A is obvious by its definition, we have only to show the following three equations to obtain the theorem.

$$F(\alpha; AQ) = F(\alpha; A)(\det Q)^k \quad (Q \in M_n), \quad (\text{A})$$

$$F(\alpha; P(g)A) = F(\alpha; A) \quad (g \in \mathfrak{S}_k^n), \quad (\text{B})$$

$$F(\alpha; \mathbb{I}_{n,k}) = \left(\frac{k!}{k^k}\right)^n f_{(k^n)}(\alpha). \quad (\text{C})$$

We introduce the two-parameter deformation of the determinant as

$$\det_{\alpha, \beta} A := \sum_{\tau, \sigma \in \mathfrak{S}_N} \alpha^{\nu(\tau)} \beta^{\nu(\sigma)} \prod_{i=1}^N a_{\tau(i)\sigma(i)}. \quad (3.1)$$

It is clear that this is symmetric in α and β , i.e. $\det_{\alpha, \beta} A = \det_{\beta, \alpha} A$. Notice that

$$F(\alpha; A) = \det_{\alpha, -1/k}(\varpi_k(A)) = \sum_{\sigma \in \mathfrak{S}_{kn}} \alpha^{\nu(\sigma)} \det_{-1/k}(\varpi_k(A)P(\sigma)).$$

Remark 3.2. The equalities (1.1) and (1.3) are readily obtained from the symmetry $\det_{\alpha, \pm 1} A = \det_{\pm 1, \alpha} A$.

3.2 Proofs of (A) and (B)

Let $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{C}^{kn}$. We have only to prove (A) when Q is an elementary matrix. Namely, it is suffice to verify

$$F(\alpha; (\mathbf{a}_1, \dots, \mathbf{a}_j + c\mathbf{a}_i, \dots, \mathbf{a}_n)) = F(\alpha; (\mathbf{a}_1, \dots, \mathbf{a}_j, \dots, \mathbf{a}_n)) \quad (i \neq j), \quad (3.2)$$

$$F(\alpha; (\mathbf{a}_1, \dots, c\mathbf{a}_j, \dots, \mathbf{a}_n)) = c^k F(\alpha; (\mathbf{a}_1, \dots, \mathbf{a}_j, \dots, \mathbf{a}_n)) \quad (c \in \mathbb{C}). \quad (3.3)$$

The equation (3.3) obviously follows from the definition of $F(\alpha; A)$ and the multilinearity of the α -determinant in column vectors. The equation (3.2) is guaranteed by the following lemma.

Lemma 3.3 (Lemma 2.3 in [1]). *Let N and k be positive integers such that $k < N$. If more than k column vectors in $A \in M_N$ are equal, then $\det_{-1/k} A = 0$.*

The second equality (B) is shown by using (2.1), (A) and the elementary fact

$$\det_{\alpha}(P(\sigma)A) = \det_{\alpha}(AP(\sigma)) \quad (A \in M_N, \sigma \in \mathfrak{S}_N) \quad (3.4)$$

as follows: For any $g \in \mathfrak{S}_k^n$, we have

$$\begin{aligned} F(\alpha; P(g)A) &= \sum_{\sigma \in \mathfrak{S}_{kn}} \alpha^{\nu(\sigma)} \det_{-1/k}(\varpi_k(P(g)A)P(\sigma)) \\ &= \sum_{\sigma \in \mathfrak{S}_{kn}} \alpha^{\nu(\sigma)} \det_{-1/k}(P(g)\varpi_k(A)P(\sigma)) \\ &= \sum_{\sigma \in \mathfrak{S}_{kn}} \alpha^{\nu(g^{-1}\sigma g)} \det_{-1/k}(\varpi_k(A)P(g)P(g^{-1}\sigma g)) = F(\alpha; A). \end{aligned}$$

3.3 Proof of (C)

Let N be a positive integer. For each partition $\lambda \vdash N$, χ^λ is the irreducible character of \mathfrak{S}_N corresponding to λ and f^λ is the number of standard tableaux with shape λ . We denote by $K_{\lambda\mu}$ the Kostka number, that is, the number of tableaux with shape λ and weight μ . Note that $f^\lambda = K_{\lambda(1^N)} = \chi^\lambda(1)$ for each $\lambda \vdash N$. By the hook formula for f^λ and the definition of $f_\lambda(x)$, we have

$$f_{(k^n)}(-1/k) = \frac{(kn)!}{k^{kn}} \frac{1}{f^{(k^n)}}, \quad f_{(k^n)}(1/n) = \frac{(kn)!}{n^{kn}} \frac{1}{f^{(k^n)}}. \quad (3.5)$$

For each pair λ, μ of partitions of N , define

$$\omega_\mu^\lambda(x) := \frac{1}{\mu!} \sum_{\tau \in \mathfrak{S}_\mu} \chi^\lambda(x\tau) \quad (x \in \mathfrak{S}_N),$$

where $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \dots$ is the Young subgroup associated to μ and $\mu! = \mu_1! \mu_2! \dots$ is its cardinality. Here we regard the i -th component \mathfrak{S}_{μ_i} in \mathfrak{S}_μ as a subgroup of \mathfrak{S}_N consisting of permutations of the μ_i letters $m+1, \dots, m+\mu_i$ with $m = \sum_{j < i} \mu_j$. It is immediate to see that ω_μ^λ is \mathfrak{S}_μ -biinvariant function on \mathfrak{S}_N . It is well known that

$$K_{\lambda\mu} = \frac{1}{\mu!} \sum_{\tau \in \mathfrak{S}_\mu} \chi^\lambda(\tau) = \omega_\mu^\lambda(1) \quad (3.6)$$

for $\lambda, \mu \vdash N$.

Let $*$ be the convolution product defined by

$$(\phi_1 * \phi_2)(x) = \sum_{\sigma \in \mathfrak{S}_N} \phi_1(x\sigma) \phi_2(\sigma^{-1})$$

for $\phi_1, \phi_2: \mathfrak{S}_N \rightarrow \mathbb{C}$. Recall that the irreducible characters satisfy

$$\chi^\lambda * \chi^\rho = \delta_{\lambda\rho} \frac{N!}{f^\lambda} \chi^\lambda \quad (\lambda, \rho \vdash N). \quad (3.7)$$

We need the following Fourier expansion formula.

Lemma 3.4 (Fourier expansion of $\alpha^{\nu(\cdot)}$).

$$\alpha^{\nu(\sigma)} = \frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\alpha) \chi^\lambda(\sigma) \quad (\sigma \in \mathfrak{S}_N). \quad (3.8)$$

By (3.7) and (3.8), we have

$$\alpha^{\nu(\cdot)} * \beta^{\nu(\cdot)} = \frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\alpha) f_\lambda(\beta) \chi^\lambda.$$

Hence it follows that

$$\det_{\alpha, \beta} X = \frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\alpha) f_\lambda(\beta) \text{Imm}_\lambda X,$$

where $\text{Imm}_\lambda X$ is the *immanant* associated to λ defined by

$$\text{Imm}_\lambda X = \sum_{\sigma \in \mathfrak{S}_N} \chi^\lambda(\sigma) x_{\sigma(1)1} x_{\sigma(2)2} \dots x_{\sigma(N)N} = \sum_{\sigma \in \mathfrak{S}_N} \chi^\lambda(\sigma) \det_0(P(\sigma^{-1})X). \quad (3.9)$$

For a partition $\mu = (\mu_1, \mu_2, \dots, \mu_l) \vdash N$, define

$$\mathbf{1}_\mu := \begin{pmatrix} \mathbf{1}_{\mu_1} & & & \\ & \mathbf{1}_{\mu_2} & & \\ & & \ddots & \\ & & & \mathbf{1}_{\mu_l} \end{pmatrix}.$$

For example, we have $\mathbf{1}_{(k^n)} = I_n \otimes \mathbf{1}_k$. We have

$$\omega_\mu^\lambda(g) = \frac{1}{\mu!} \text{Imm}_\lambda(P(g)\mathbf{1}_\mu)$$

since

$$\det_0(P(g)\mathbf{1}_\mu) = \begin{cases} 1 & g \in \mathfrak{S}_\mu, \\ 0 & g \notin \mathfrak{S}_\mu. \end{cases}$$

Thus it follows that

$$\det_{\alpha,\beta}(P(g)\mathbf{1}_\mu) = \frac{\mu!}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\alpha) f_\lambda(\beta) \omega_\mu^\lambda(g) \quad (g \in \mathfrak{S}_N). \quad (3.10)$$

As a particular case, we have

$$\det_{\alpha,\beta}(I_n \otimes \mathbf{1}_k) = \frac{(k!)^n}{(kn)!} \sum_{\lambda \vdash kn} f^\lambda K_{\lambda,(k^n)} f_\lambda(\alpha) f_\lambda(\beta). \quad (3.11)$$

by putting $N = kn$, $\mu = (k^n)$ and $g = 1$ because of (3.6).

Now we further assume that $\beta = -1/k$ in (3.11). By definition, we have $f_\lambda(-1/k) = 0$ unless $\lambda_1 \leq k$. On the other hand, $K_{\lambda,(k^n)} = 0$ unless $l(\lambda) \leq n$. Therefore, the summand in (3.11) vanishes unless $\lambda = (k^n)$. Thus it follows that

$$F(\alpha; \mathbb{I}_{n,k}) = \det_{\alpha,-1/k}(I_n \otimes \mathbf{1}_k) = \frac{(k!)^n}{(kn)!} f^{(k^n)} f_{(k^n)}(\alpha) f_{(k^n)}(-1/k) = \frac{(k!)^n}{k^{kn}} f_{(k^n)}(\alpha),$$

where we use (3.5) in the last equality. This completes the proof of the theorem.

4 Proofs of the lemmas

Here we prove Lemmas used in the previous section. The proof of Lemma 3.1 below is different from the one given in [1], and is rather elementary. The proof of Lemma 3.3 is just a revision of the one given in [1]. Lemma 3.4 is prove by using Okounkov-Vershik theory [7] on representations of symmetric groups.

4.1 Proof of Lemma 3.1

Let $f: M_{kn,n} \rightarrow \mathbb{C}$ be a function satisfying the conditions (W1)–(W3). We put

$$\phi(\sigma) = f(P(\sigma)\mathbb{I}_{n,k})$$

for $\sigma \in \mathfrak{S}_{kn}$. Notice that

$$\phi(\tau\sigma\tau') = \phi(\sigma) \quad (\tau, \tau' \in \mathfrak{S}_k^n, \sigma \in \mathfrak{S}_{kn})$$

by (W3) and the invariance $P(\tau')\mathbb{I}_{n,k} = \mathbb{I}_{n,k}$. Let I and J be fixed complete systems of representatives of the coset $\mathfrak{S}_{kn}/\mathfrak{S}_k^n$ and the double coset $\mathfrak{S}_k^n \backslash \mathfrak{S}_{kn} / \mathfrak{S}_k^n$ respectively, and define $I(\sigma)$ for each $\sigma \in J$ to be the subset of I such that $\coprod_{\tau \in I(\sigma)} \tau \mathfrak{S}_k^n = \mathfrak{S}_k^n \sigma \mathfrak{S}_k^n$. Notice that $\phi(\tau) = \phi(\sigma)$ for each $\tau \in I(\sigma)$.

By (W1), we have

$$f(A) = \sum_{1 \leq j_1, \dots, j_{kn} \leq n} a_{1j_1} \dots a_{kn,j_{kn}} f({}^t(e_{j_1} \dots e_{j_{kn}}))$$

for $A = (a_{ij}) \in M_{kn,n}$, where e_1, e_2, \dots, e_n are the standard basis vectors of \mathbb{C}^n . By (W2), $f({}^t(e_{j_1} \dots e_{j_{kn}}))$ vanishes unless the matrix rank of $(e_{j_1} \dots e_{j_{kn}})$ equals n , or (j_1, \dots, j_{kn}) is a permutation of $(\overbrace{1, \dots, 1}^k, \dots, \overbrace{n, \dots, n}^k)$. Hence it follows that

$$f(A) = \sum_{\tau \in I} \phi(\tau) a_{\tau(1)1} \dots a_{\tau(kn)n} = \sum_{\sigma \in J} \phi(\sigma) \sum_{\tau \in I(\sigma)} a_{\tau(1)1} \dots a_{\tau(kn)n}.$$

If we take $A = \mathbb{I}_{n,k}X$, $X = (x_{ij}) \in M_n$, then we have

$$f(\mathbb{I}_{n,k}X) = \sum_{\sigma} \phi(\sigma) \sum_{\tau \in I(\sigma)} x^{M(\tau)} = \sum_{\sigma} \phi(\sigma) \#I(\sigma) x^{M(\sigma)},$$

where

$$x^{M(\sigma)} = \prod_{i,j} x_{ij}^{m_{ij}(\sigma)}, \quad m_{ij}(\sigma) = \#\{s \mid (i-1)k < s \leq ik, (j-1)k < \sigma(s) \leq jk\}.$$

Notice that $M(\sigma)$ depends only on the double coset $\mathfrak{S}_k^n \sigma \mathfrak{S}_k^n$. On the other hand, by (W2), we have $f(\mathbb{I}_{n,k}X) = f(\mathbb{I}_{n,k})(\det X)^k$. Hence we get

$$\phi(\sigma) = \frac{f(\mathbb{I}_{n,k})}{\#I(\sigma)} \times \text{coefficient of } x^{M(\sigma)} \text{ in } (\det X)^k. \quad (4.1)$$

As a result, we have

$$f(A) = \frac{f(\mathbb{I}_{n,k})}{\text{wrdet}_k \mathbb{I}_{n,k}} \text{wrdet}_k A$$

as desired.

4.2 Proof of Lemma 3.3

It is enough to prove that the sum (1.1) is divisible by $(1+\alpha) \dots (1+(k-1)\alpha)$. We see that (1.1) is equal to

$$\sum_{\tau \in \mathfrak{S}_N} \left(\sum_{\sigma \in \mathfrak{S}_k} \alpha^{\nu(\tau\sigma)} \right) \prod_{i=1}^N a_{\tau(i)i}.$$

For each $\tau \in \mathfrak{S}_N$, there uniquely exists $\tau_0 \in \mathfrak{S}_k$ such that $\nu(\tau\sigma) = \nu(\tau\tau_0^{-1}) + \nu(\tau_0\sigma)$ for any $\sigma \in \mathfrak{S}_k$. In fact, if we define g_i and τ_i for $i = n, n-1, \dots, 1$ recursively by

$$\tau_n = \tau; \quad g_i = (i \ \tau_i(i)), \quad \tau_{i-1} = g_i \tau_i,$$

then we have $\tau_0 = g_k g_{k-1} \dots g_1$. It follows that

$$\sum_{\sigma \in \mathfrak{S}_k} \alpha^{\nu(\tau\sigma)} = \alpha^{\nu(\tau\tau_0^{-1})} \det_{\alpha} \mathbf{1}_k = \alpha^{\nu(\tau\tau_0^{-1})} (1+\alpha) \dots (1+(k-1)\alpha).$$

Hence the sum (1.1) is divisible by $(1+\alpha) \dots (1+(k-1)\alpha)$.

4.3 Proof of Lemma 3.4

Let X_1, \dots, X_N be the Jucys-Murphy elements of the group algebra $\mathbb{C}\mathfrak{S}_N$:

$$X_k = (1 \ k) + (2 \ k) + \dots + (k-1 \ k) \quad (1 \leq k \leq N).$$

It is elementary to see that

$$\phi := \sum_{\sigma \in \mathfrak{S}_N} \alpha^{\nu(\sigma)} \sigma = (1 + \alpha X_1)(1 + \alpha X_2) \dots (1 + \alpha X_N),$$

which is central since ν is a class function. So it is a linear combination of the projections

$$P_{\lambda} := \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \chi^{\lambda}(\sigma) \sigma \quad (\lambda \vdash N).$$

For each partition $\lambda \vdash N$, let $\{v_T\}_{T \in \text{STab}(\lambda)}$ be the Gelfand-Tsetlin basis (or Young basis) of the irreducible representation \mathbf{S}^{λ} of \mathfrak{S}_N associated to λ , where $\text{STab}(\lambda)$ is the set of standard tableaux with shape λ . It is known that if the number written in the (i, j) -position of T is k , then

$$X_k v_T = (j - i) v_T.$$

Hence it follows that

$$\phi v_T = \prod_{(i,j) \in \lambda} (1 + (j-i)\alpha) v_T = f_\lambda(\alpha) v_T$$

for any $T \in \text{STab}(\lambda)$. Thus we have

$$\text{Tr } \phi|_{\mathcal{S}^\lambda} = f^\lambda f_\lambda(\alpha),$$

so that we get

$$\phi = \sum_{\lambda \vdash N} \text{Tr } \phi|_{\mathcal{S}^\lambda} P_\lambda = \sum_{\sigma \in \mathfrak{S}_N} \left(\frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\alpha) \chi^\lambda(\sigma) \right) \sigma$$

as desired.

5 Corollaries of the discussion

We obtain the following “determinantal” formula of the values of ω_μ^λ and the Kostka numbers $K_{\lambda\mu}$ for rectangular-shaped Young diagrams λ as a byproduct of the discussion above.

Corollary 5.1. *For any $g \in \mathfrak{S}_{kn}$ and $\mu \vdash kn$, it holds that*

$$\omega_\mu^{(k^n)}(g) = \frac{f^{(k^n)}}{\mu!} \frac{\det_{-1/k, 1/n}(P(g)\mathbf{1}_\mu)}{\det_{-1/kn} \mathbf{1}_{kn}}. \quad (5.1)$$

In particular, it holds that

$$K_{(k^n)\mu} = \frac{f^{(k^n)}}{\mu!} \frac{\det_{-1/k, 1/n} \mathbf{1}_\mu}{\det_{-1/kn} \mathbf{1}_{kn}}. \quad (5.2)$$

Proof. We first notice that $\det_{-1/kn} \mathbf{1}_{kn} = (kn)!/(kn)^{kn}$ by (2.3). Putting $N = kn$, $\alpha = -1/k$ and $\beta = 1/n$ in (3.10), we have

$$\det_{-1/k, 1/n}(P(g)\mathbf{1}_\mu) = \frac{\mu!}{(kn)!} \sum_{\lambda \vdash kn} f^\lambda f_\lambda(-1/k) f_\lambda(1/n) \omega_\mu^\lambda(g).$$

Since $f_\lambda(-1/k) = 0$ unless $\lambda_1 \leq k$ and $f_\lambda(1/n) = 0$ unless $l(\lambda) \leq n$, the summand in the right-hand side of the equation above vanishes unless $\lambda = (k^n)$. Therefore we have

$$\begin{aligned} \det_{-1/k, 1/n}(P(g)\mathbf{1}_\mu) &= \frac{\mu!}{(kn)!} f^{(k^n)} f_{(k^n)}(-1/k) f_{(k^n)}(1/n) \omega_\mu^{(k^n)}(g) \\ &= \frac{\mu!}{f^{(k^n)}} \frac{(kn)!}{(kn)^{kn}} \omega_\mu^{(k^n)}(g) = \frac{\mu!}{f^{(k^n)}} \det_{-1/kn}(\mathbf{1}_{kn}) \omega_\mu^{(k^n)}(g), \end{aligned}$$

which implies (5.1). The equation (5.2) is readily obtained by putting $g = 1$ in (5.1). \square

By putting $\mu = (1^{kn})$ in Corollary 5.1, we have a formula of irreducible characters for rectangular diagrams.

Corollary 5.2. *It holds that*

$$\frac{\chi^{(k^n)}(g)}{f^{(k^n)}} = \frac{\det_{-1/k, 1/n} P(g)}{\det_{-1/kn} \mathbf{1}_{kn}} \quad (5.3)$$

for $g \in \mathfrak{S}_{kn}$. \square

Remark 5.3. Assume that $m \leq N = kn$. Let $\iota: \mathfrak{S}_m \rightarrow \mathfrak{S}_N$ be the natural inclusion. Then we have

$$\frac{\det_{\alpha, \beta} P(\iota(w))}{\det_{\alpha, \beta} I_N} = \frac{\det_{\alpha, \beta} P(w)}{\det_{\alpha, \beta} I_m} = \sum_{\sigma \in \mathfrak{S}_m} \alpha^{\nu(w\sigma)} \beta^{\nu(\sigma^{-1})}.$$

Hence, for $w \in \mathfrak{S}_m$, the formula (5.2) gives Stanley’s formula [6]

$$\frac{N!}{(N-m)!} \frac{\chi^{(k^n)}(\iota(w))}{\chi^{(k^n)}(1)} = (-1)^m \sum_{\sigma \in \mathfrak{S}_m} (-k)^{\kappa(w\sigma)} n^{\kappa(\sigma^{-1})}, \quad (5.4)$$

where $\kappa(\sigma)$ denotes the number of disjoint cycles in σ .

We look at another particular case where $\mu = (k^n)$. For each $\lambda \vdash N$, we put $\omega^\lambda := \omega_\lambda^\lambda$.

Corollary 5.4. *Let n, k be positive integers. For any $g \in \mathfrak{S}_{kn}$,*

$$\omega^{(k^n)}(g) = \frac{\text{wrdet}_k(P(g)\mathbb{I}_{n,k})}{\text{wrdet}_k \mathbb{I}_{n,k}}$$

holds.

Proof. As we see in the proof of Corollary 5.1, we have

$$\det_{-1/k, 1/n}(P(g)\mathbf{1}_{(k^n)}) = \frac{(k!)^n}{(kn)!} f_{(k^n)}(1/n) \omega^{(k^n)}(g) = \omega^{(k^n)}(g) f_{(k^n)}(1/n) \text{wrdet}_k \mathbb{I}_{n,k}.$$

On the other hand, by Theorem 2.2, we have

$$\det_{-1/k, 1/n}(P(g)\mathbf{1}_{(k^n)}) = \det_{-1/k, 1/n}(\varpi_k(P(g)\mathbb{I}_{n,k})) = f_{(k^n)}(1/n) \text{wrdet}_k(P(g)\mathbb{I}_{n,k}).$$

Combining these two, we obtain the desired conclusion. \square

Remark 5.5. By Corollary 5.4 and (4.1), we have

$$\omega^{(k^n)}(\sigma) = \frac{\text{coefficient of } x^{M(\sigma)} \text{ in } (\det X)^k}{|\mathfrak{S}_k^n : \mathfrak{S}_k^n \cap \sigma^{-1} \mathfrak{S}_k^n \sigma|}$$

for $\sigma \in \mathfrak{S}_{kn}$.

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